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SYNTHETIC CHARACTERIZATION OF REDUCED ALGEBRAS

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1. Introduction and statement of main results

We pursue in this note the line of thought that the toposes one constructs in algebraic and differential geometry provide a basis on which a simple-minded synthetic reasoning can take place and be made useful, as the development of synthetic differential geometry shows. But such reasoning also pertains to pure algebraic geometry; here, we may quote [3] as an early pilot project. The present note considers an 'internal-infinitary' analogue of a main result in there, about the generic local ring R, which lives in the Zariski topos $\mathcal{Z}$. In loc. cit. we proved (Proposition 2.2) that for $R \in \mathcal{Z}$ we have (for any natural number n)

$$(*) \qquad \forall (r_1, \dots, r_n) \in R^n: \quad \neg \left(\bigwedge_{i=1}^{n} (r_i = 0) \right) \Rightarrow \bigvee_{i=1}^{n} (r_i \text{ invertible}).$$

We now ask: when does this result hold if the externally indexed family $r_1, \dots, r_n$ is replaced by an 'internally indexed family' $\{r_x \mid x \in M\}$ where M is an object of $\mathcal{Z}$. So we ask: when do we have

$$(**) \qquad \vdash \forall f \in R^M: \quad \neg (f \equiv 0) \Rightarrow \exists x \in M: \; f(x) \text{ is invertible.}$$

(Here, "$f \equiv 0$" is short for "$\forall x \in M: f(x) = 0$".) We give an answer for the case where $\mathcal{Z} = \mathcal{Z}_k$, meaning the Zariski topos over an algebraically closed field k (= the classifying topos for local k-algebras), and where M is affine, i.e. represented by some finite type k-algebra A; we write $M = \bar{A}$. Note that $R = \overline{k[x]}$. The result then is

Theorem 1. *The principle* (**) *holds for* $M = \bar{A}$ *if and only if the* k*-algebra* A *is reduced.*

Recall that "A reduced" means: 0 is the only nilpotent element in A. By Hilbert Nullstellensatz, this is equivalent to saying: if an element $g \in A$ is killed by all $A \to k$, then $g = 0$.

In [4, Theorem III.10.1], we proved that (*) holds for the "Dubuc topos $\mathscr{G}^{\text{op}}$", where $\mathscr{G}$ is the category of germ-determined $\mathbb{T}_\infty$-algebras A (i.e. $A = C^\infty(\mathbb{R}^n)/I$

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where the ideal I is of local character, Dubuc [2]), and with $R=\overline{C^\infty(\mathbb{R})}$. Also here, we can say exactly for which representables $M=\bar{A}$ $(A\in\mathscr{B})$ (**) holds:

Theorem 2. *The principle (**) holds for $M=\bar{A}$ if and only if the $\mathbb{T}_\infty$-algebra A is point-determined.*

Recall from [4, Definition III.5.9] that "A point-determined" means that if an element $g\in A$ is killed by all $A\to\mathbb{R}$, then $g=0$.

The proofs of these two theorems are closely related; we give the proof for the Zariski-topos case, with parenthetical remarks about modifications for the Dubuc-topos case.

2. Generalities concerning the toposes

The Zariski topos over an algebraically closed field k and the Dubuc topos have some interesting common features. We let $\mathscr{E}$ denote either of them and otherwise keep the notation of Section 1. We let $\mathscr{B}$ denote the site of definition of either.

In these sites, the trivial algebra $\{0\}$ occurs, and is the only object covered by the empty family. The non-trivial algebras A in the two sites have a common feature: There exists a k-algebra map $A\to k$ (respectively, there exists a $\mathbb{T}_\infty$-algebra map $A\to\mathbb{R}$). The former fact follows again from Hilbert Nullstellensatz (the objects of $\mathscr{B}$ being finitely presented k-algebras), and the latter is immediate from the definition of "germ-determined" (in fact, Dubuc [2] motivated us to invent this notion). We refer to both these facts as "Nullstellensatz".

Now the following proposition is almost immediate, and more or less well known.

Proposition 2.1. *The terminal object $\mathbb{1}$ of $\mathscr{E}$ has no proper subobjects.*

Proof. The initial object $\emptyset$ of $\mathscr{E}$ is given as the functor which to each non-trivial algebra in $\mathscr{B}$ associates the empty set, and to the trivial algebra associates a one-point set. Now assume that U is a subobject of the terminal object, and that $U\neq\emptyset$. Then for some non-trivial $B\in\mathscr{B}$, we have $U(B)\neq\emptyset$. But B being non-trivial, there exists by Nullstellensatz some $B\to k$ (respectively $B\to\mathbb{R}$). This means that there exist maps $\mathbb{1}\to\bar{B}$ and $\bar{B}\to U$, thus also $\mathbb{1}\to U$. Since U is a subobject of $\mathbb{1}$, $U=\mathbb{1}$.

Proposition 2.2. *Let $h:\bar{E}\to R$ be a map from a representable object into R, such that, for any point $p:\mathbb{1}\to\bar{E}$, $h\circ p$ is an invertible (global) element of R. Then h itself is an invertible (generalized) element of R, i.e. factors through the subobject $\operatorname{Inv}(R)\to R$.*

Proof. In the site $\mathscr{B}$, the assumption expresses that $h\in E$ has the property that any $p:E\to k$ (respectively $p:E\to\mathbb{R}$) takes h into a non-zero element. But then h must be invertible in E, for otherwise, $E/(h)$ would be a non-trivial algebra in $\mathcal{M}$ (for the Dubuc-topos case, the fact that $E \in \mathcal{M} \Rightarrow E/(h) \in \mathcal{M}$ follows from [4, Theorem III.6.3]), and thus there would, by Nullstellensatz, exist some $E/(h) \to k$ (respectively $E/(h) \to \mathbb{R}$), whose composite with $E \to E/(h)$ would take h to 0, contrary to assumption.

3. Proof of the theorems

Let $A \in \mathcal{M}$ be reduced (respectively point-determined), and let

$$f \in_{\bar{B}} R^{\bar{A}}$$

(where $B \in \mathcal{M}$). The exponential adjoint of $f: \bar{B} \to R^{\bar{A}}$ is a map $\bar{B} \times \bar{A} \to R$ which we denote $\hat{f}$. Now R and the terminal object are representable. The full subcategory of $\mathcal{E}$ consisting of representable (= affine) objects is closed under finite inverse limits, so that $\hat{f}^{-1}(0)$ is an affine subobject $\bar{C}$ of $\bar{B} \times \bar{A}$. In fact, $\hat{f}$ corresponds to an element $f^{\vee}$ in the algebra $B \otimes_k A$, and $\bar{C}$ is represented by $C = B \otimes_k A/(f^{\vee})$. (For the Dubuc topos case, $B \otimes_k A$ is replaced by $(B \otimes_{\infty} A)^{\wedge}$, using notation of [4, III §§5,6].)

We consider the largest subobject S of $\bar{B}$ so that $\vdash_S \forall x \in \bar{A}: f(x) = 0$. So

$$S = [\![y \in \bar{B} \mid \forall x \in \bar{A}: \hat{f}(y,x) = 0]\!]$$

$$= [\![y \in \bar{B} \mid \forall x \in \bar{A}: (y,x) \in \bar{C}]\!] = \forall_{\pi} \bar{C}$$

where $\pi: \bar{B} \times \bar{A} \to \bar{B}$ is the projection, and $\forall_{\pi}$ is "right adjoint to pulling-back along π". The assumption $\vdash_{\bar{B}} \neg (f \equiv 0)$, i.e.

$$\vdash_{\bar{B}} \neg (\forall x \in \bar{A}: f(x) = 0)$$

is thus equivalent to $\forall_{\pi} \bar{C} = \emptyset$.

Consider now an arbitrary point of $\bar{B}$, i.e. a map $y: \mathbb{1} \to \bar{B}$, and consider $\pi^{-1}(y) \subseteq \bar{B} \times \bar{A}$. Then we do not have $\pi^{-1}(y) \subseteq \bar{C}$, for this would be equivalent to $y \subseteq \forall_{\pi} \bar{C}$ which equals $\emptyset$. So the inclusion

$$\pi^{-1}(y) \cap \bar{C} \subseteq \pi^{-1}(y) \tag{3.1}$$

defines a proper subobject of $\pi^{-1}(y)$. Identifying $\pi^{-1}(y)$ with $\bar{A}$, this subobject sits in the pull-back

$$\begin{array}{ccc}
\pi^{-1}(y) \cap \bar{C} & \longrightarrow & \mathbb{1} \times \bar{A} \cong \bar{A} \\
\downarrow & & \downarrow {\scriptstyle y \times \bar{A}} \\
\bar{C} & \longrightarrow & \bar{B} \times \bar{A}
\end{array} \quad ;$$

since affines are closed under finite inverse limits, $\pi^{-1}(y)\cap\bar{C}$ is affine, and comes from a pushout in $\mathscr{B}$

$$\begin{array}{ccc} E & \twoheadleftarrow & A \\ \uparrow & & \uparrow \\ C & \twoheadleftarrow & B\otimes_k A \end{array}$$

for some E which is a quotient algebra of A, since C is a quotient algebra of $B\otimes_k A$. (For the Dubuc topos case: replace $\otimes_k$ by $\otimes_\infty$.) Since (3.1) is a proper subobject, the kernel I of $A\to E$ is non trivial, and since A is reduced (respectively point-determined), there is some $x: A\to k$ (respectively $A\to\mathbb{R}$) with $x(I)\neq 0$, i.e. which does not factor over $A\to E$. This x represents a point $x:\mathbb{1}\to\bar{A}$ which does not factor across $\pi^{-1}(y)\cap\bar{C}$. We have thus proved: to every $y:\mathbb{1}\to\bar{B}$, there is some $x:\mathbb{1}\to\bar{A}$ with (y,x) not in $\bar{C}$, or equivalently, with

$$\mathbb{1}\xrightarrow{(y,x)}\bar{B}\times\bar{A}\xrightarrow{\hat{f}}R \tag{3.2}$$

different from $0\in_{\mathbb{1}}R$.

Since $\hom_{\mathscr{E}}(\mathbb{1},R)=k$ (respectively $=\mathbb{R}$), this means that the element (3.2) is actually invertible.

Let the algebras A and B be presented as

$$k[X_1,\dots,X_n]/I \quad\text{and}\quad k[Y_1,\dots,Y_m]/J$$

respectively (in the Dubuc topos case, replace $k[X_1,\dots,X_n]$ by $C^\infty(\mathbb{R}^n)$, etc.). These presentations induce, in $\mathscr{E}$, inclusions

$$\bar{A}\rightarrowtail R^n,\qquad \bar{B}\rightarrowtail R^m.$$

The $\hat{f}:\bar{B}\times\bar{A}\to R$ we consider, is represented by $f^\vee\in B\otimes_k A$ (respectively $\in B\otimes_\infty A$), and we pick $F^\vee\in k[Y_1,\dots,Y_m,X_1,\dots,X_n]$ (respectively $\in C^\infty(\mathbb{R}^{m+n})$) that maps to $f^\vee$ by the canonical $k[Y_1,\dots,X_n]\to B\otimes_k A$ (respectively ...).

Synthetically in $\mathscr{E}$, this means that we have a commutative diagram:

$$\begin{array}{ccc} \bar{B}\times\bar{A} & \xrightarrow{\hat{f}} & R \\ \downarrow & \nearrow_{\hat{F}} & \\ R^m\times R^n & & \end{array}$$

We may identify $\hom_{\mathscr{E}}(\mathbb{1},R^n)$ with k^n (respectively $\mathbb{R}^n$), and $\hom_{\mathscr{E}}(\mathbb{1},\bar{A})$ with $Z(I)\subseteq k^n$ (respectively $Z(I)\subseteq\mathbb{R}^n$), the zero set of the ideal I. Similarly for $\bar{B}$: $\hom(\mathbb{1},\bar{B})=Z(J)\subseteq k^m$ (respectively $\subseteq\mathbb{R}^m$).

The result above now implies: to every $y \in Z(J)$, there exist $x \in Z(I)$ so that $\hat{F}(y, x)$ is invertible. For each $y \in Z(J)$, we *pick* one such x and denote it $x(y)$.

We can now construct a Zariski-open covering of k^m (respectively, an "ordinary" open covering of $\mathbb{R}^m$); it consists of U_0 = complement of $Z(J)$, and of

$$U_y = \{ y' \in k^m \mid \hat{F}(y', x(y)) \text{ is invertible} \}$$

for every $y \in Z(J) = \hom(\mathbb{1}, \bar{B})$, (respectively, ...).

By construction of $x(y), y \in U_y$, so the sets U_0 and the U_y's do cover k^m (respectively $\mathbb{R}^m$), and are open. They provide a (co-) cover in $\mathscr{B}$ of $k[Y_1, \dots, Y_m]$ (respectively of $C^\infty(\mathbb{R}^m)$) with respect to the Zariski Grothendieck-topology on $\mathscr{B}^{op}$ (respectively with respect to Dubuc's Grothendieck topology on $\mathscr{B}^{op}$ [4, Definition III.7.2]).

We push this co-cover out along $k[Y_1, \dots, Y_m] \twoheadrightarrow B$ (respectively $C^\infty(\mathbb{R}^m) \twoheadrightarrow B$) to get a co-cover in $\mathscr{B}$ of B

$$\{ B \to B_y \mid y \in Z(J) \}$$

(note that U_0 pushes out to the trivial algebra, which is co-covered by the empty family, so may be dropped).

For each $y \in Z(J)$, consider the map in $\mathscr{E}$

$$\tilde{x}_y = \left(\bar{B}_y \to \bar{B} \to \mathbb{1} \xrightarrow{x(y)} \bar{A} \right). \tag{3.3}$$

For every $y' : \mathbb{1} \to \bar{B}$ which factors through $\bar{B}_y$, we have, by construction of U_y and hence of B_y, that $f(y', x(y))$ is invertible. So the composite map $\hat{f}(-, \tilde{x}_y) : \bar{B}_y \to R$ has the property that it takes any $\mathbb{1} \to \bar{B}_y$ to an invertible $\mathbb{1} \to R$. From Proposition 2.2 it follows that $\hat{f}(-, \tilde{x}_y)$ factors through $\mathrm{Inv}(R) \to R$, or, equivalently, that the (generalized) element $\tilde{x}_y$ in (3.3) of $\bar{A}$ (defined at stage $\bar{B}_y$) has

$$\vdash_{\bar{B}_y} f(\tilde{x}_y) \text{ is invertible.}$$

Thus the covering $\{ \bar{B}_y \to \bar{B} \mid y \in \hom(\mathbb{1}, \bar{B}) \}$ and the elements $\tilde{x}_y$ witness validity of

$$\vdash_{\bar{B}} \exists x : f(x) \text{ is invertible.}$$

This proves one implication in Theorem 1 and 2. Conversely, if $\bar{A}$ is such that (**) holds for $M = \bar{A}$, it is easy to see that A is reduced (respectively point determined). For, let $f \in A$ be killed by all $x : A \to k$ (respectively $x : A \to \mathbb{R}$). In $\mathscr{E}$, f represents a map $\bar{A} \to R$, i.e.

$$f \in_{\mathbb{1}} R^{\bar{A}}.$$

Consider $C = A/(f)$. Then

$$\bar{C} = [\![a \in \bar{A} \mid f(a) = 0]\!] \subseteq \bar{A}.$$

Now consider $V_\pi \bar{C} \subseteq \mathbb{1}$. By Proposition 2.1, either $V_\pi \bar{C} = \emptyset$ or $V_\pi \bar{C} = \mathbb{1}$. In the former case. $\vdash_{\mathbb{1}} \neg (f \equiv 0)$, so by assumption, $\vdash_{\mathbb{1}} \exists x : f(x)$ is invertible. This is contrary to the assumption that f is killed by all $x: A \to k$ (respectively $\to \mathbb{R}$). In the case $\forall_\pi \bar{C} = \mathbb{1}$, we get $\bar{C} = \bar{A}$, so $f = 0$. This proves the converse implication in Theorems 1 and 2.

4. Applications

Both the toposes studied here have R as a model of synthetic differential geometry. In particular, there exists a differentiation process

$$\frac{\partial}{\partial x} : R^R \to R^R.$$

The theorems proved hold for $M = R$ since $R = \overline{k[x]}$ (respectively $R = \overline{C^\infty(\mathbb{R})}$) which is reduced (respectively point determined). We then have

Corollary 4.1. *We have*

$$\vdash_{\mathbb{1}} \forall f \in R^R : \neg\left(\frac{\partial f}{\partial x} \equiv 0\right) \Rightarrow \exists x : f(x) \text{ is invertible.}$$

Proof. It suffices, by the theorems, to prove $\neg(f \equiv 0)$, i.e. to derive a contradiction from $f \equiv 0$. But $f \equiv 0$ implies, by the rules of differentiation $\partial/\partial x$ that $\partial f/\partial x \equiv 0$. This contradicts $\neg(\partial f/\partial x) \equiv 0$.

Corollary 4.2. *For any reduced (respectively point-determined) A, the apartness relation $\#$ on $R^{\bar{A}}$ given by*

$$f \# g \quad \text{iff} \quad \neg(f \equiv g)$$

is separated, i.e. satisfies

$$f \# g \ \Rightarrow \ (h \# f) \vee (h \# g) \quad \forall f, g, h.$$

Proof. Assume $f \# g$, i.e. $\neg(f \equiv g)$. By Theorems 1 and 2 $\exists x$ with $f(x) - g(x)$ invertible in R. Since R is a local ring $f(x) - h(x)$ or $h(x) - g(x)$ must be invertible. In the former case, $h \# f$, in the latter, $h \# g$.

We finally present a problem related to the theorem: if we replace the condition "$\neg(f \equiv 0)$" by "$\neg(f$ factors through $\Delta)$" (where $\Delta \subseteq R$ is the subobject $\neg\neg\{0\}$), for which M is then the conclusion "$\exists x \in M : f(x)$ is invertible" valid?

Acknowledgement

The present research was inspired by Bunge's [1], where she attempts the beginn-

ings of a synthetic functional analysis, in the sense that she investigates neighbour and apartness relations, i.e. some weak kind of topology, on function space objects like R^R, in the Dubuc topos.

References

[1] M. Bunge, Synthetic aspects of C^∞-mappings, J. Pure Appl. Algebra 28 (1983) 41-63.

[2] E. Dubuc, C^∞-schemes, Amer. J. Math. 103 (1981) 683-690.

[3] A. Kock, Universal projective geometry via topos theory, J. Pure Appl. Algebra 9 (1976) 1-24.

[4] A. Kock, Synthetic Differential Geometry, London Math. Soc. Lecture Note Series 51 (Cambridge Univ. Press, Cambridge, 1981).